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The influence of a layer of mud on the train of waves generated by a moving pressure distribution

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Abstract. Two-dimensional free surface flows generated by a moving distribution of pressure are considered. The bottom is assumed to be covered by a thin layer of mud. The mud is modelled as a viscous fluid. The problem is solved numerically by a boundary integral equation method. It is shown that the layer of mud produces a damping of the waves in the far field. Profiles of the free surface and of the surface of the mud are presented.

1. Introduction

The problem of the generation of waves by a moving distribution of pressure has been considered by many investigators (see for example Lamb [1], Schwartz [2] and Vanden-Broeck and Dias [3]). In these calculations it is assumed that the fluid is of infinite depth or bounded below by a rigid horizontal bottom.

In this paper we look at the effect of replacing the rigid horizontal bottom by a thin layer of mud. This problem is motivated by the observation that a layer of mud can have drastic effects on wave damping and on the motion and wave resistance of a ship [4–7].

Our configuration consists of two superposed layers of fluid in a channel. The lower layer represents the mud which is modelled as a viscous fluid. The upper layer is inviscid and bounded above by a free surface. The pressure distribution is assumed to move at a constant velocity on the free surface. We approximate the flow in the lower layer by using the lubrication equations. This enables us to formulate the problem as an inviscid flow with appropriate boundary conditions. The lubrication equations were used recently by King, Tuck and Vanden-Broeck [8] to study waves on a thin layer of fluid. The inviscid problem is then reformulated as a boundary integral equation. This equation is discretized and the resultant algebraic equations are solved by Newton's method. The numerical procedure is similar to the one used by many investigators to study purely inviscid free surface flows (see, for example, Vanden-Broeck and Tuck [9], Schwartz [2], Forbes [10] and Vanden-Broeck and Dias [3]). The idea of reducing a problem with a thin viscous layer to an inviscid one with appropriate boundary conditions was used before by Kit and Shemer [11].

Our results show that the layer of mud produces a damping of the waves. Since the waves are ultimately of small amplitude in the far field, the rate of damping can be evaluated by using a linear theory. It is found that the rate of damping approaches zero for very small or very large values of the Reynolds number. Values of the wave resistance are also presented.

The problem is formulated in Section 2. The integral equation is derived in Section 3. The numerical procedure is described in Section 4 and the results are presented in Section 5.

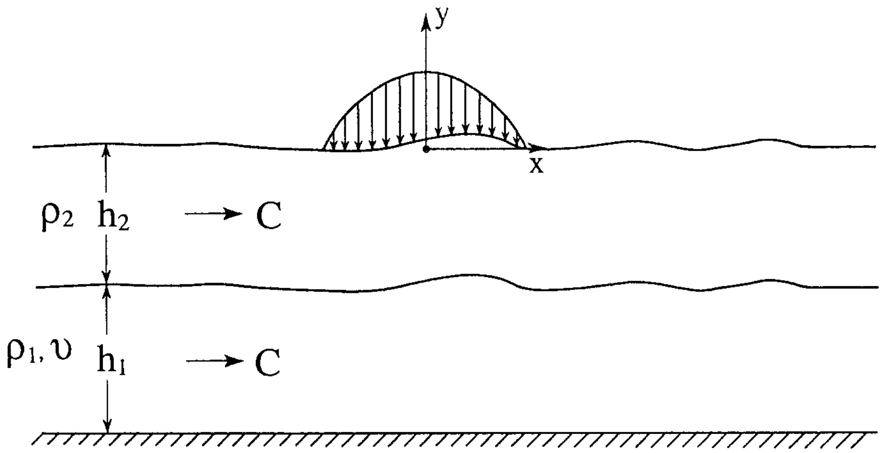

Fig. 1. Sketch of the flow and of the coordinates.

2. Mathematical formulation

We consider a channel with two superposed layers of fluids. The flow is bounded below by a horizontal bottom and above by a free surface (see Figure 1). The lower layer is an incompressible and viscous fluid of density ρ_1, and viscosity ν. The upper layer is an incompressible and inviscid fluid of density ρ_2. We are interested in the train of waves generated by a prescribed distribution of pressure moving at a constant velocity c on the free surface. We take a frame of reference moving with the pressure distribution so that the flow is steady. Far upstream the lower and upper layers are characterized by constant thickness h_1 and h_2 and a constant velocity c. We introduce Cartesian coordinates with the x-axis along the level of the free surface far upstream and the y-axis directed vertically upwards. Gravity is acting along the negative y-axis. We describe the shape of the free surface by $y = \eta_F(x)$ and the shape of the interface between the two layers by $y = \eta_I(x)$. The functions $\eta_F(x)$ and $\eta_I(x)$ have to be found as part of the solution.

We first approximate the flow in $-h_1 - h_2 < y < \eta_I(x)$ by assuming that the lower layer is thin. If L denotes a length scale for changes in the x-direction, prescribed for example by the wavelength of the oscillations on the interface $y = \eta_I(x)$, we define

$$\epsilon = h_1/L \tag{2.1}$$

and assume $\epsilon \ll 1$. This is the situation that applies in lubrication theory [12] and allows asymptotic simplification of the Navier–Stokes equations as $\epsilon \to 0$. In order that gravity and the pressure exerted by the upper layer on $y = \eta_I(x)$ have an effect, we must demand that the fluid pressure p has the lubrication pressure scale

$$p = O(\rho\nu c L^{-1}\epsilon^{-2}). \tag{2.2}$$

We must also demand that the Reynolds number

$$R_L = cL/\nu \tag{2.3}$$

be not too large; specifically that

$$R_L \ll \epsilon^{-2}. \tag{2.4}$$

The Navier–Stokes equations then simplify to

$$
\begin{aligned}
&u_x + v_y = 0 \\
&0 = -\frac{1}{\rho_1} p_x + \nu u_{yy} \\
&0 = -\frac{1}{\rho_1} p_y - g.
\end{aligned}
\tag{2.5}
$$

Here u and v are the x and y components of the velocity.

In the upper layer the flow is irrotational. Therefore the complex velocity $u - c - iv$ is an analytic function of $z = x + iy$ in the strip $\eta_I(x) < y < \eta_F(x)$.

The equations are to be solved subject to the boundary conditions

(i) $u = c, v = 0$ on $y = -h_1 - h_2$,
(ii) $u_y = 0$ on $y = \eta_I^-(x)$,
(iii) $p(x, \eta_I^-(x)) = p(x, \eta_I^+(x))$,
(iv) $v = u\eta_{I_x}$ on $y = \eta_I^\pm(x)$,
(v) $v = u\eta_{F_x}$ on $y = \eta_F(x)$,
(vi) $\dfrac{(u^2 + v^2)}{2} + g\eta_F(x) + \dfrac{P_A(x)}{\rho_2} = \dfrac{c^2}{2}$ on $y = \eta_F(x)$
(vii) $u \to c, v \to 0$ as $x \to -\infty$

Here η_I^- and η_I^+ denote the lower and upper sides of the interface η_I. The conditions (ii), (iii), (iv) on the interface $y = \eta_I(x)$ represent, respectively, continuity of tangential stress, continuity of the pressure and the kinematic conditions. In condition (iii) we have neglected the normal viscous stress. This is consistent with the lubrication approximation. The conditions (v) and (vi) on the free surface $y = \eta_F(x)$ are the kinematic condition and the dynamic boundary condition. The function $P_A(x)$ in (vi) is the prescribed distribution of pressure. We assume that $P_A(x)$ has compact support, i.e. that $P_A(x)$ vanishes outside some finite interval. The condition (vii) is the radiation condition which prohibits waves far upstream. It is the same condition as used in inviscid theory.

This concludes the formulation of the problem. We seek the functions $u(x, y)$, $v(x, y)$, $p(x, y)$, $\eta_F(x)$ and $\eta_I(x)$ satisfying (2.5) in $-h_1 - h_2 < y < \eta_I(x)$, the boundary conditions (i)–(vii) and such that $u - iv$ is an analytic function of $z = x + iy$ in $\eta_I(x) < y < \eta_F(x)$.

3. Reformulation as an integro-differential equation

We introduce dimensional variables by choosing c as the reference velocity and h_2 as the reference length. In the remaining part of the paper all the variables are assumed to be dimensionless. As we shall see, the problem is then characterized by the Froude number

$$F = c/(gh_2)^{1/2}, \tag{3.1}$$

the Reynolds number

$$R = ch_2/\nu, \tag{3.2}$$

the depth ratio

$$\beta = h_1/h_2, \tag{3.3}$$

and the density ratio

$$\delta = \rho_1/\rho_2 > 1. \tag{3.4}$$

The quantity $w(z) = u - iv - 1$ is an analytic function of $x + iy$ in $\eta_I(x) < y < \eta_F(x)$. Hence by Cauchy's theorem, when z is on the free surface

$$w(z) = -\frac{1}{\pi i} \oint_C \frac{w(\zeta)\,\mathrm{d}\zeta}{\zeta - z} \tag{3.5}$$

with a Cauchy principal-value interpretation. Here C denotes the free surface and the interface between the two layers.

We denote the values of u and v on the free surface and on the upper and lower sides of the interface by $u_F(x)$, $v_F(x)$, $u_I^+(x)$, $v_I^+(x)$, $u_I^-(x)$ and $v_I^-(x)$, respectively. Taking the real part of (3.5), we obtain, after some algebra

$$\begin{aligned}
&\pi[u_F(x) - 1] \\
&= -\int_{-\infty}^{+\infty} \frac{[u_F(s) - 1 + v_F(s)\eta_F'(s)][\eta_F(x) - \eta_F(s)] + [(u_F(s) - 1)\eta_F'(s) - v_F(s)](s - x)}{(s - x)^2 + (\eta_F(s) - \eta_F(x))^2} \mathrm{d}s \\
&\quad + \int_{-\infty}^{+\infty} \frac{[u_I^+(s) - 1 + v_I^+(s)\eta_I'(s)][\eta_F(x) - \eta_I(s)] + [(u_I^+(s) - 1)\eta_I'(s) - v_I^+(s)](s - x)}{(s - x)^2 + (\eta_I(s) - \eta_F(x))^2} \mathrm{d}s
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
&\pi[u_I^+(x) - 1] \\
&= -\int_{-\infty}^{+\infty} \frac{[u_F(s) - 1 + v_F(s)\eta_F'(s)][\eta_I(x) - \eta_F(s)] + [(u_F(s) - 1)\eta_F'(s) - v_F(s)](s - x)}{(s - x)^2 + (\eta_F(s) - \eta_I(x))^2} \mathrm{d}s \\
&\quad + \int_{-\infty}^{+\infty} \frac{[u_I^+(s) - 1 + v_I^+(s)\eta_I'(s)][\eta_I(x) - \eta_I(s)] + [(u_I^+(s) - 1)\eta_I'(s) - v_I^+(s)](s - x)}{(s - x)^2 + (\eta_I(s) - \eta_I(x))^2} \mathrm{d}s.
\end{aligned} \tag{3.7}$$

The equations (2.5) with the boundary conditions (i), (ii), (iv) and (vii) are easily integrated and the resulting single equation put in the form (see Appendix)

$$\frac{\mathrm{d}p^-}{\mathrm{d}x} + \frac{\eta_I'(x)}{F^2} = \frac{3}{R} \frac{\eta_I(x) + 1}{(\eta_I(x) + \beta + 1)^3}. \tag{3.8}$$

Here we use the notation $p^-(x) = p(x, \eta_I^-(x))$ and $p^+(x) = p(x, \eta_I^+(x))$. Using Bernoulli's equation on the upper side of the interface, we obtain

$$p^+(x) = 1 - \frac{u^2 + v^2}{2} - \frac{\eta_I(x)}{F^2}. \tag{3.9}$$

Substituting (iii) and (3.9) into (3.8), yields

$$\frac{\delta - 1}{F^2}\eta_I'(x) + \frac{3}{R}\frac{\eta_I(x) + 1}{(\eta_I(x) + \beta + 1)^3} + \delta(uu_x + vv_x) = 0. \tag{3.10}$$

Finally, (iv) gives

$$v_I^+(x) = u_I(x)\eta_I'(x). \tag{3.11}$$

For given values of F, R, β and δ, relations (vi), (3.6), (3.7), (3.10) and (3.11) define a system of integro-differential equations for the four unknown functions $u_F(x)$, $\eta'_F(x)$, $u_I^+(x)$ and $\eta'_I(x)$. This system is solved numerically in the next section. In all the calculations we chose

$$P_A(x) = 0, \quad \text{for} \quad |x| > b \tag{3.12}$$

and

$$P_A(x) = \epsilon \exp\left(\frac{b^2}{x^2 - b^2}\right) \quad \text{for} \quad |x| < b. \tag{3.13}$$

Here ϵ and b are two constants which define the strength and the length of the pressure distribution.

After a solution has been computed the wave resistance or drag D can be evaluated by the integral

$$D = \int_0^\infty P_A(x)\eta'_F(x)\,\mathrm{d}x. \tag{3.14}$$

4. Numerical procedure

We seek a numerical solution of the nonlinear integro-differential system (vi), (3.6), (3.7), (3.10) and (3.11). We define $N+1$ mesh points on the free surface $y = \eta_F(x)$ and $N+1$ mesh points on the upper side of the interface $y = \eta_I^+(x)$ by specifying values $x = X_K$, where

$$X_K = -\frac{EN}{2} + E(K-1), \quad K = 1, \ldots, N+1. \tag{4.1}$$

here E is the interval of discretization. We shall also make use of the intermediate mesh points $X_{K+1/2} = (X_{K+1} + X_K)/2$, $K = 1, 2, \ldots, N$.

We now define the $4N + 4$ corresponding fundamental unknown quantities

$$u_{F_K} = u_F(X_K), \quad K = 1, 2, \ldots, N+1 \tag{4.2}$$

$$\eta'_{F_K} = \eta'_F(X_K), \quad K = 1, 2, \ldots, N+1 \tag{4.3}$$

$$u_{I_K} = u_I^+(X_K), \quad K = 1, 2, \ldots, N+1 \tag{4.4}$$

and

$$\eta'_{I_K} = \eta'_I(X_K), \quad K = 1, 2, \ldots, N+1. \tag{4.5}$$

We directly obtain the values of $v_{F_K} = v_F(X_K)$ and of $v_{I_K} = v_I^+(X_K)$ in terms of the unknowns by using (v) and (3.11). We estimate the values of $\eta_{F_K} = \eta_F(X_K)$ and $\eta_{I_K} = \eta_I(X_K)$ in terms of the fundamental unknowns by the trapezoidal rule with $\eta_F(1) = 0$ and $\eta_I(1) = -1$.

We then evaluate the values of $u_F(x)$, $v_F(x)$, $u_I(x)$, $v_I(x)$, $u'_F(x)$, $v'_F(x)$, $u'_I(x)$ and $v'_I(x)$ in terms of the unknowns by four-point difference formulas. We satisfy (vi) and (3.10) at the intermediate mesh points. This yields $2N$ nonlinear equations.

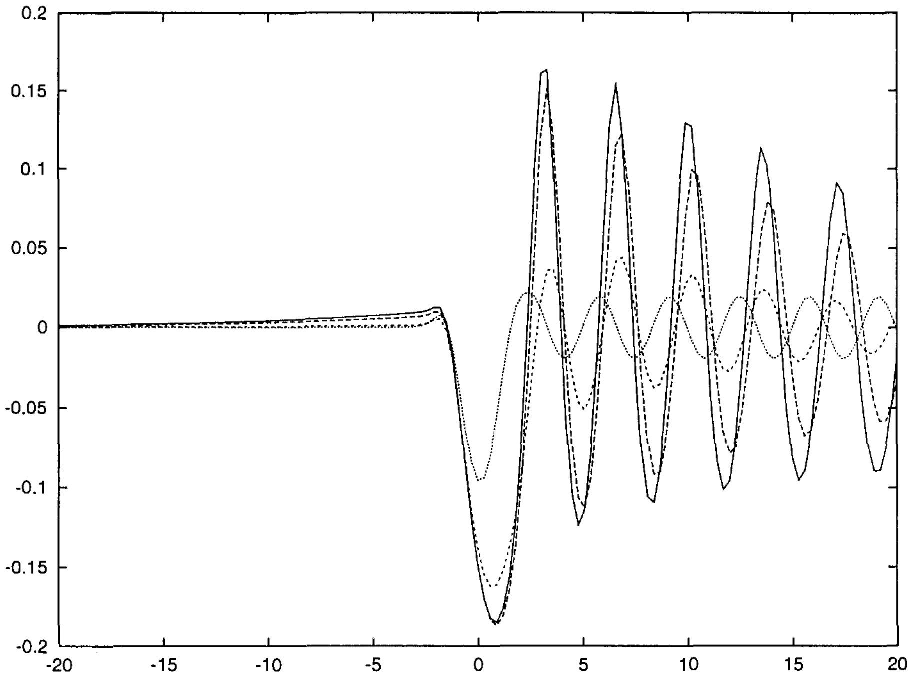

Fig. 2. Free surface profiles for $F = 0.72$, $b = 0.5$, $\delta = 1$, $\beta = 0.25$, $\epsilon = 0.5$ and $R = 9$ (——), $R = 15$ (– – –), $R = 60$ (- - - -), $R = \infty$ (……).

Next we satisfy (3.6) and (3.7) at the point $x = X_{K+1/2}$, $K = 1, 2, \ldots, N$ by applying the trapezoidal rule to (3.6) and (3.7), with a sum over the points $x = X_J$, $J = 1, 2, \ldots, N+1$. The symmetry of the discretization and of the trapezoidal rule with respect to the singularity of the integrand at $s = x$ enables us to evaluate this Cauchy principal value integral by ignoring the singularity, with an accuracy no less than a nonsingular integral. This yields $2N$ extra nonlinear equations.

Four more equations are obtained by imposing the far field conditions

$$\eta'_{I_1} = 0 \tag{4.6}$$

$$\eta'_{F_1} = 0 \tag{4.7}$$

$$u_{F_1} = 1 \tag{4.8}$$

$$u_{I_1} = 1. \tag{4.9}$$

We now have $4N + 4$ equations for the $4N + 4$ unknowns (4.2)–(4.5). This system is solved by Newton's method for given values of F, R, δ, β, ϵ and b.

5. Discussion of results

We used the numerical scheme of Section 4 to compute solutions for $\delta = 1$ and various values of F, β, R, ϵ and b. Solutions for different values of F, β and b were found to be qualitatively similar. Therefore we assume in the remaining part of the paper that $b = 0.5$, $\beta = 0.25$ and $F = 0.72$.

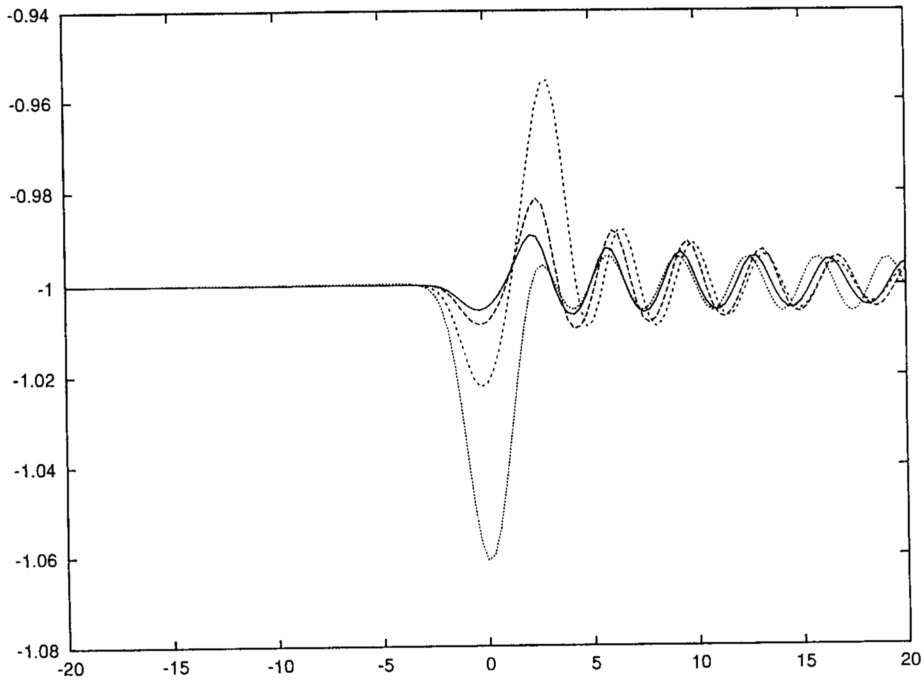

Fig. 3. Profiles of the surface of the mud for $F = 0.72$, $b = 0.5$, $\delta = 1$, $\beta = 0.25$, $\epsilon = 0.5$ and $R = 9$ (——), $R = 15$ (– – –), $R = 60$ (- - - -), $R = \infty$ (······).

Typical profiles of the free surface and of the surface of the surface of the mud are shown in Figures 2 and 3 for $\epsilon = 0.5$ and various values of R. As $R \to 0$, the surface of the mud becomes increasingly flat and the waves on the free surface approach a train of waves of constant amplitude as $x \to \infty$. For $0 < R \leqslant \infty$, there are waves on both the free surface and the surface of the mud. For $0 < R < \infty$, the amplitude of the waves on the free surface and on the surface of the mud decays to zero as $x \to \infty$. For $R = \infty$, the waves approach again a train of waves of constant amplitude as $x \to \infty$. The problem with $R = \infty$ is completely inviscid and describes waves in a two fluid system when the pressure in the lower layer is assumed to be hydrostatic (see (3.8) with $R = \infty$). Similar two-fluid systems were considered before by Olmstead and Raynor [13], Tuck [14] and Vanden-Broeck [15].

For $0 < R < \infty$, the waves are ultimately of small amplitude as $x \to \infty$. Therefore we can evaluate the rate of decay in the far field by using linear theory. For this purpose we write

$$u = 1 + u^* \tag{5.1}$$

$$v = v^* \tag{5.2}$$

$$\eta_F = \eta_F^* \tag{5.3}$$

$$\eta_I = \beta + \eta_I^* \tag{5.4}$$

where the variables with * denote small perturbations.

Substituting (5.1)–(5.4) into (3.10), (3.11) and (vi) and retaining only the terms linear in u^*, v^*, η_I^* and η_F^* yields

$$\frac{3}{R\beta^3} v^* + u^*_{xx} = 0 \quad y = -1 \tag{5.5}$$

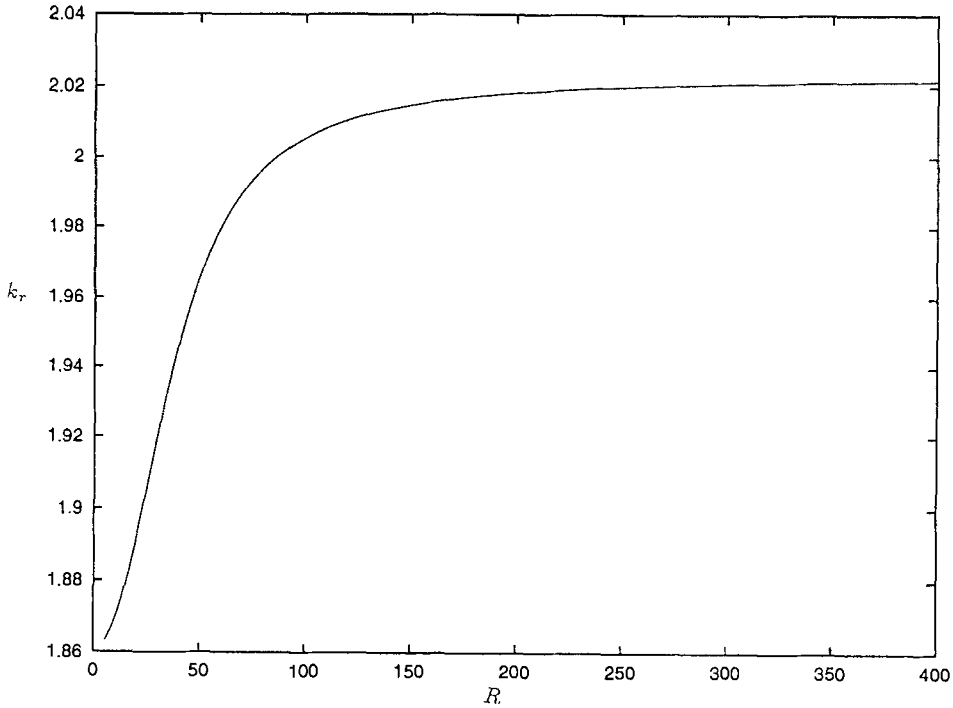

Fig. 4. Value of the wave number k_r predicted by the linear theory versus R for $F = 0.72$, $\delta = 1$ and $\beta = 0.25$.

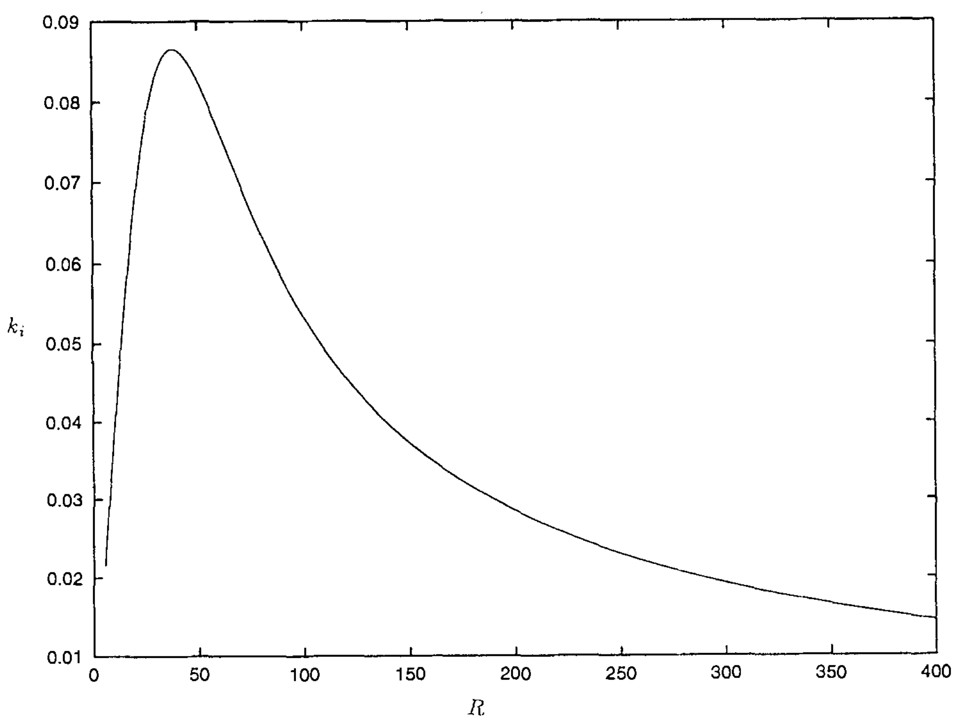

Fig. 5. Value of the rate of decay k_i predicted by the linear theory versus R for $F = 0.72$, $\delta = 1$ and $\beta = 0.25$.

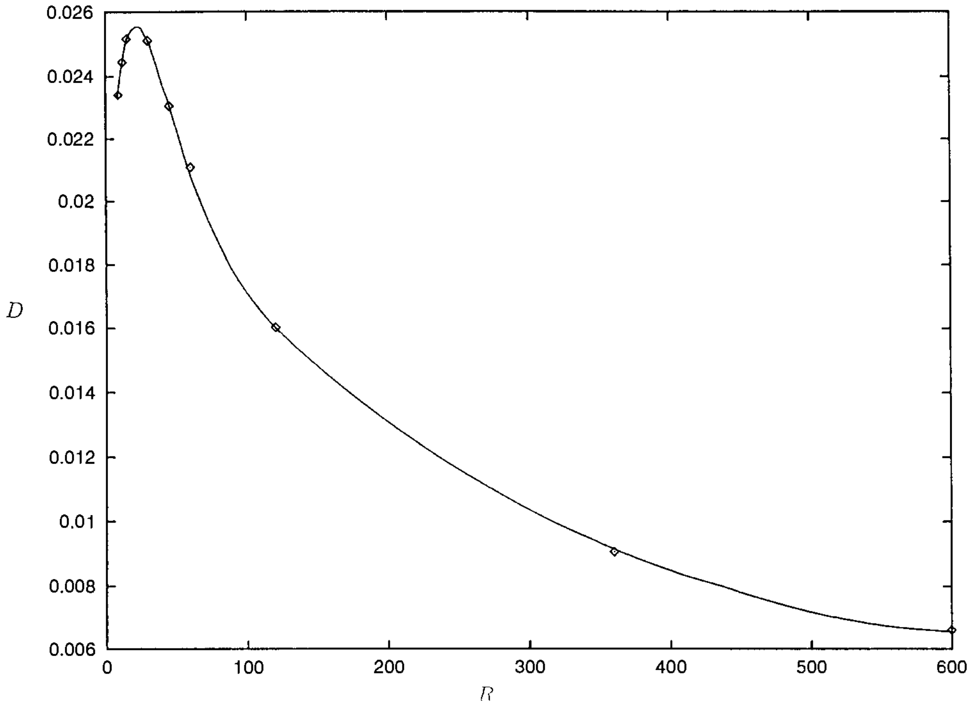

Fig. 6. Values of the drag D versus R for $F = 0.72$, $\delta = 1$, $\beta = 0.25$ and $\epsilon = 0.5$.

$$u_x^* + \frac{1}{F^2}v^* = 0 \quad y = 0. \tag{5.6}$$

In Figure 6, we present values of the drag D versus R for $\epsilon = 0.5$. The results show that D is a decreasing function of R for sufficiently large values of R.

Next we use the irrotationality of the flow in the upper layer to write

$$u^* = \phi_x^*, \quad v^* = \phi_y^*. \tag{5.7}$$

Since we are interested in the rate of decay of the waves, it is sufficient to assume

$$\phi^* = e^{ikx}[Ae^{ky} + Be^{-ky}]. \tag{5.8}$$

Here A, B and k are constants. The constant k is in general complex. Its real part, k_r, is the wave number and its imaginary part, k_i, is the rate of decay.

To evaluate k we substitute (5.7) and (5.8) into (5.5) and (5.6). This leads to the following linear system for A and B

$$\left[\frac{3}{R\beta^3}e^{-k} - ik^2e^{-k}\right]A + \left[-\frac{3}{R\beta^3}e^{k} - ik^2e^{k}\right]B = 0 \tag{5.9}$$

$$\left[-k + \frac{1}{F^2}\right]A - \left[k + \frac{1}{F^2}\right]B = 0. \tag{5.10}$$

In order for (5.9)–(5.10) to have a nontrivial solution for A and B, we require that the determinant of the coefficients vanishes. This leads to:

$$e^{-k}\left[-\frac{3}{R\beta^3}\left(k + \frac{1}{F^2}\right) + ik^3 + \frac{ik^2}{F^2}\right] + e^{k}\left[-\frac{3}{R\beta^3}\left(k - \frac{1}{F^2}\right) - ik^3 + \frac{ik^2}{F^2}\right] = 0. \tag{5.10}$$

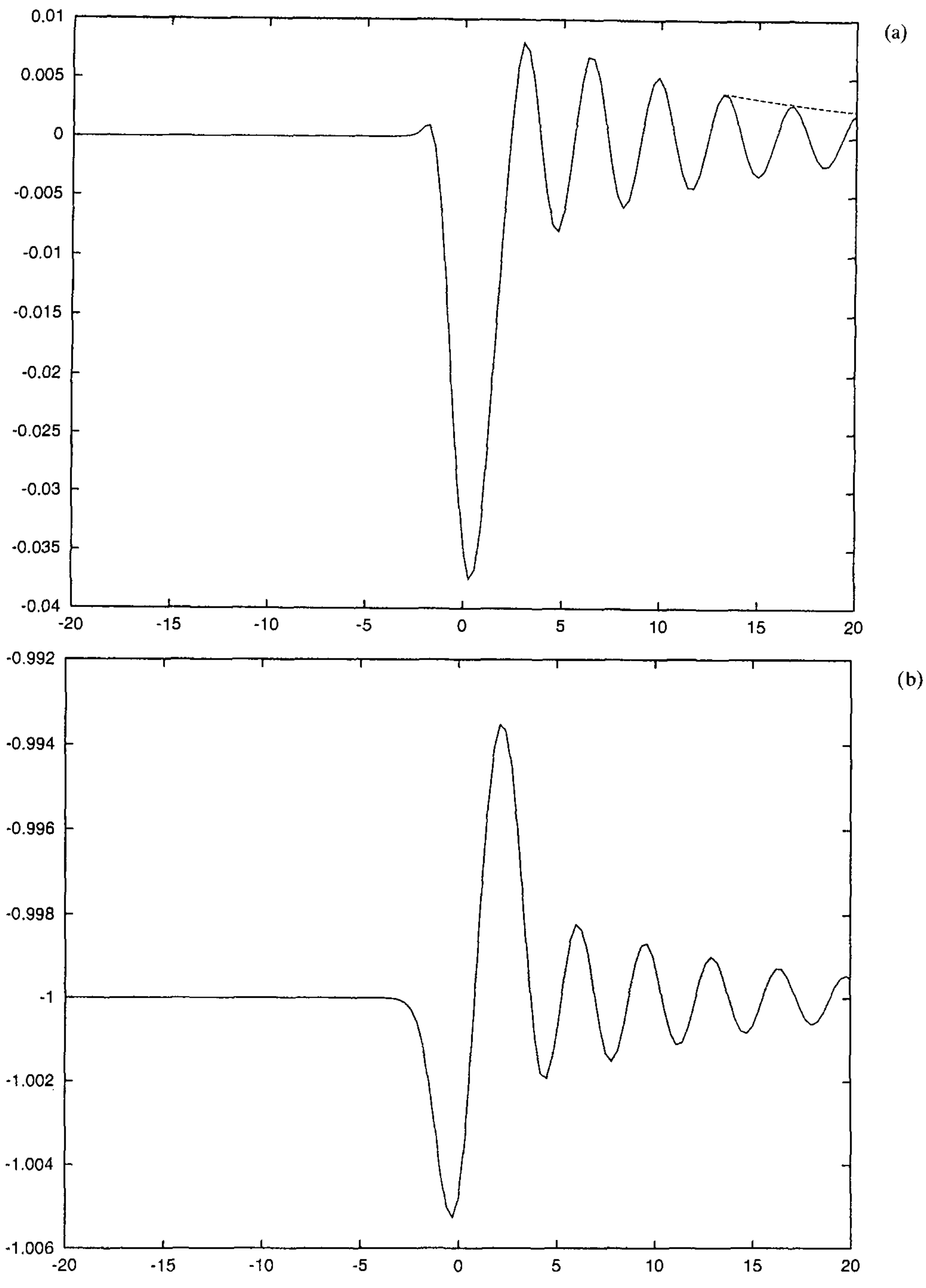

Fig. 7. Free surface profile (a) and profile of the surface of the mud (b) for $F = 0.72$, $b = 0.5$, $\delta = 1$, $\beta = 0.25$, $R = 60$ and $\epsilon = 0.125$.

For $R = 0$, k is real and (5.10) reduces to the classical dispersion relation of linear water waves in water of finite depth

$$F^2 = \frac{\tanh k}{k}. \tag{5.11}$$

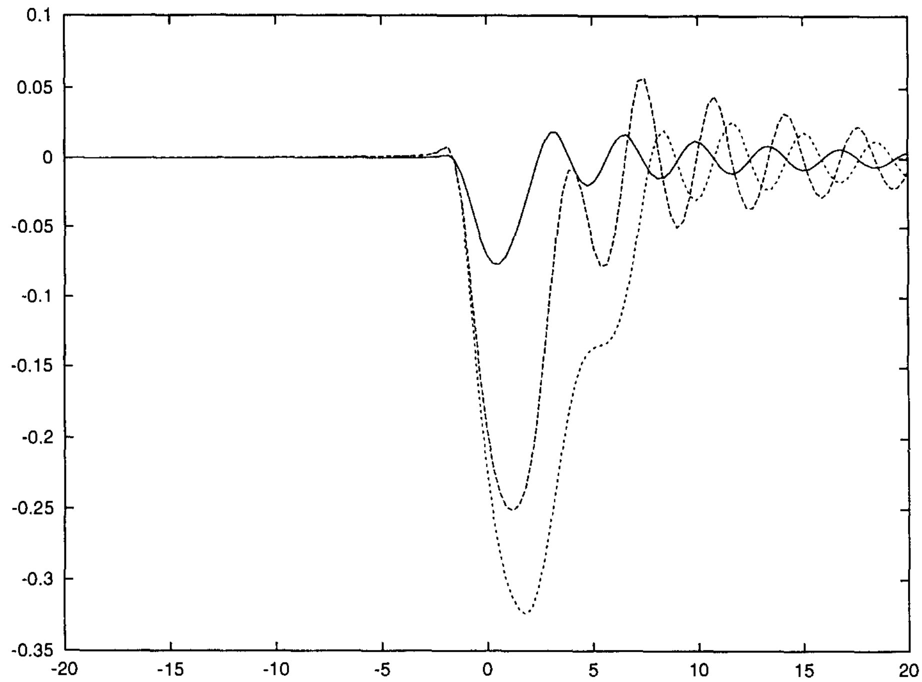

Fig. 8. Free surface profiles for $F = 0.72, b = 0.5, \delta = 1, \beta = 0.25, R = 60$ and $\epsilon = 0.25$ (——), $\epsilon = 0.7$ (– – –), $\epsilon = 0.8$ (- - - -).

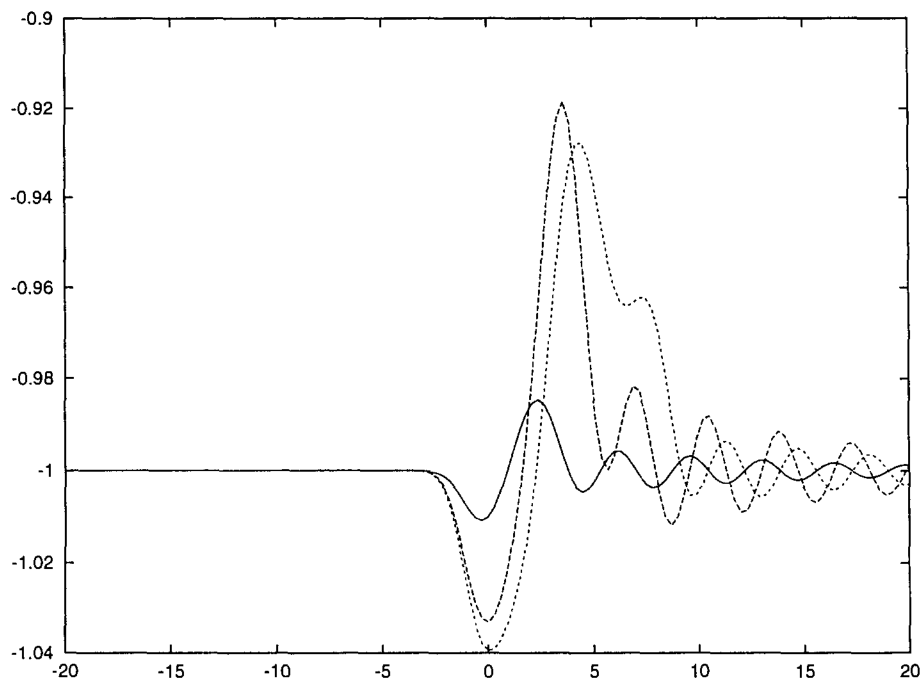

Fig. 9. Profiles of the surface of the mud for $F = 0.72, b = 0.5, \delta = 1, \beta = 0.25, R = 60$ and $\epsilon = 0.25$ (——), $\epsilon = 0.7$ (– – –), $\epsilon = 0.8$ (- - - -).

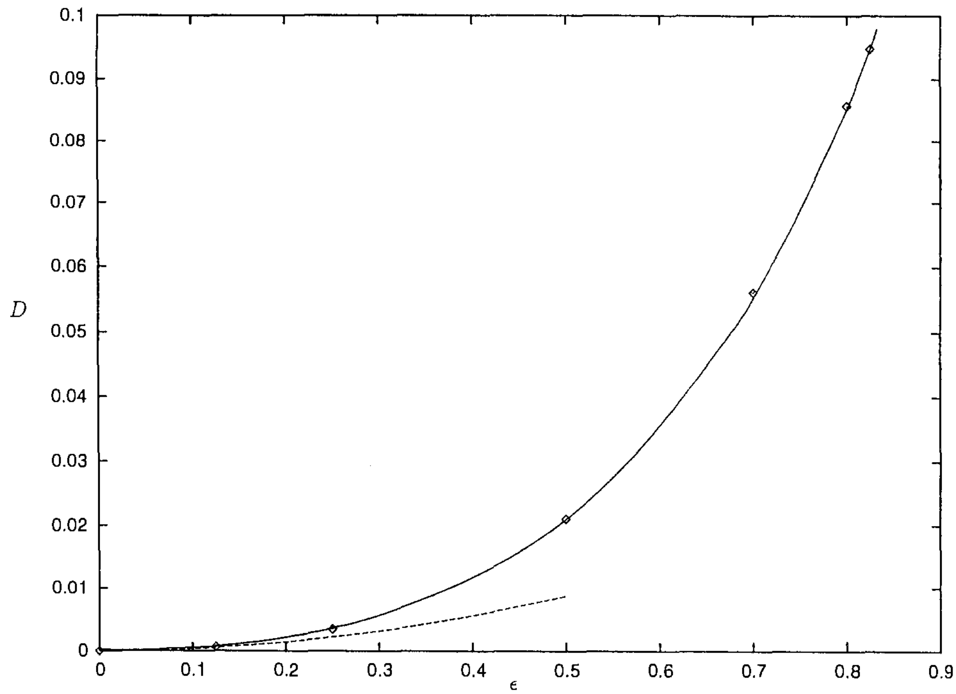

Fig. 10. Values of the drag D versus ϵ for $F = 0.72$, $\delta = 1$, $\beta = 0.25$ and $R = 60$. The broken curve corresponds to linear theory.

For $R = \infty$, k is also real and (5.10) reduces to

$$\frac{1}{F^2} = k \tanh k. \tag{5.12}$$

Equation (5.12) is the linear dispersion relation for linear waves in a two-fluid system when the pressure in the lower layer is hydrostatic.

For $0 < R < \infty$, k is complex and (5.10) has to be solved numerically. This can easily be done by Newton's method. Numerical values of k_r and k_i are shown in Figures 4 and 5 as functions of the Reynolds number R. Figure 5 shows that k_i approaches zero for both very small and very large values of R.

Next we look at the effect of increasing ϵ on the solutions. Profiles of the free surface and of the surface of the mud are shown in Figures 2, 3, 7–9 for $R = 60$ and various values of ϵ. For small values of ϵ, the amplitude of the waves on the free surface and on the surface of the mud increases proportionally to ϵ. This behavior can be expected since the flow can be linearized about the uniform stream for ϵ small. As ϵ increases nonlinear effects become predominant and the amplitude of the waves first increases with ϵ and then decreases. In Figure 10, we show the values of D versus ϵ. The broken curve corresponds to the linear theory. It was obtained by running one Newton iteration with the uniform stream as the initial guess. Figure 10 shows that the nonlinear results deviate from the linear theory for $\epsilon > 0.25$.

Finally, we compare the rate of decay predicted by the linear theory (see Figures 4 and 5) with our numerical solution of Figure 7a. We first use the linear theory and find $k_i = 0.076$ for $R = 60$. Next we write $\eta_I^l(x) = Ge^{-k_i x}$ and evaluate the constant G by fitting a point of the numerical profile of Figure 7a for large x. The curve $\eta_I^l(x)$ is shown in Figure 7a by the broken curve. The rate of decay is seen to be in good agreement with the numerical profile.

6. Conclusions

We have shown that a layer of mud can produce a damping of the train of waves generated by a disturbance moving at the surface of a layer of fluid. Our model is based on two assumptions for the layer of mud. First we neglected the rheological properties of the mud and described it as a Newtonian fluid. Secondly we approximated the Navier Stokes equations by the lubrication equations. These assumptions enabled us to solve the problem by a boundary integral equation method.

Acknowledgment

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Appendix: Derivation of (3.8)

To describe the layer of mud, it is convenient to introduce temporally coordinates with the origin on the bottom. Thus we define $Y = y + h_1 + h_2$ and $h(x) = \eta_I(x) + h_1 + h_2$.

Integrating the third equation in (2.5) with respect to y gives

$$p = -\rho_1 g Y + f(x). \tag{A1}$$

Here $f(x)$ is an arbitrary function of x. Substituting (A1) into the second equation (2.5), integrating two times with respect to y and using the boundary conditions (i) and (ii) yields

$$\nu u = \frac{1}{\rho_1} f'(x) \frac{Y^2}{2} - \frac{1}{\rho_1} f'(x) h(x) Y + \nu c. \tag{A2}$$

Differentiating (A2) with respect to x, substituting in the first equation (2.5), integrating with respect to y and using (i) gives

$$v = -\frac{1}{\rho_1 \nu} f''(x) \frac{Y^3}{6} + \frac{1}{\rho_1 \nu} f''(x) h(x) \frac{Y^2}{2} + \frac{1}{\rho_1 \nu} f'(x) h'(x) \frac{Y^2}{2} \tag{A3}$$

Substituting (A3) into (iv), after some algebra and an integration with respect to x, gives

$$\frac{1}{3} f'(x) h^3(x) = \rho_1 \nu c h(x) + C. \tag{A4}$$

Here C is a constant of integration.

Using (vii), we find $C = -\rho_1 \nu c h_1$ and the equation (A4) becomes (3.8) when rewritten in dimensionless variables.

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